

1 From zero to infinity: minimum to maximum
2 diversity of the planet by spatio-parametric
3 Rao's quadratic entropy

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48 **Abstract**

49 **Aim:** The majority of work done to gather information on Earth di-
50 versity has been carried out by in-situ data, with known issues related
51 to epistemology (e.g., species determination and taxonomy), spatial
52 uncertainty, logistics (time and costs), among others. An alternative
53 way to gather information about spatial ecosystem variability is the
54 use of satellite remote sensing. It works as a powerful tool for attaining
55 rapid and standardized information. Several metrics used to calculate
56 remotely sensed diversity of ecosystems are based on Shannon’s In-
57 formation Theory, namely on the differences in relative abundance of
58 pixel reflectances in a certain area. Additional metrics like the Rao’s
59 quadratic entropy allow the use of spectral distance beside abundance,
60 but they are point descriptors of diversity, namely they can account
61 only for a part of the whole diversity continuum. The aim of this
62 paper is thus to generalize the Rao’s quadratic entropy by proposing
63 its parameterization for the first time.

64 **Innovation:** The parametric Rao’s quadratic entropy, coded in R, i)

65 allows to represent the whole continuum of potential diversity indices
66 in one formula, and ii) starting from the Rao's quadratic entropy, al-
67 lows to explicitly make use of distances among pixel reflectance values,
68 together with relative abundances.

69 **Main conclusions:** The proposed unifying measure is an integra-
70 tion between abundance- and distance-based algorithms to map the
71 continuum of diversity given a satellite image at any spatial scale.

72 *Keywords:* biodiversity; ecological informatics; modelling; remote sens-
73 ing; satellite imagery.

74

75 1 Introduction

76 Since Alexander von Humboldt (1769-1859), the spatial component of nature
77 has played a relevant role in natural science. In the development of theoretical
78 and empirical models in ecology, spatial structure represents a key concept to
79 allow scientists to link ecological patterns to the generating processes and to
80 the functional networking among organisms (Borcard and Legendre, 2002).

81 The majority of the work done to gather information about Earth diver-
82 sity has been carried out by in-situ data, with known issues related to epis-
83 temology (e.g., species determination and taxonomy), spatial uncertainty,
84 logistics (time and costs), among others (Rocchini et al., 2011).

85 Using satellite remote sensing can at least help attaining rapid and stan-
86 dardized information about Earth diversity (Gillespie, 2005; Rocchini et al.,
87 2005). Furthermore, remote sensing can also be used to monitor some ecosys-
88 tem functions and parameters such as temperatures, photosynthesis, vegeta-
89 tion biomass production and precipitation (Schimel et al., 2019; Zellweger et
90 al., 2019) that can be useful to define the different niches of in-situ species,
91 following first Goodall (1970) ideas, who envisaged future diversity measures
92 as those based on niche theory (Hutchinson, 1959). The free access to re-
93 mote sensing data (see Zellweger et al., 2019) has opened new ways to study
94 ecosystem diversity and biodiversity issues (Rocchini et al., 2013). The spec-
95 tral data related to pixels, as operational geographical units, are descriptions
96 of pieces of land that allow us to define a new kind of Earth “diversity”,
97 which may complement in-situ biodiversity measurement.

98 Diversity varies with area, thus investigating multiple spatial grains, until
99 wide extents, is important to effectively monitor spatial diversity change in
100 space and time (MacArthur et al., 1966). This is especially true in macroe-
101 cology, where the primary aim is to model large-scale spatial patterns to infer
102 the ecological processes which generated them, particularly considering the
103 recent effect of global changes worldwide (Hobohm et al., 2019). In order to
104 determine the horizontal distribution of diversity within a satellite image (i.e.
105 which areas within the image are more diverse than others), diversity indices
106 are usually spatially referenced by calculating the index within a moving
107 window.

108 Several metrics that measure diversity from satellites rely on the Shan-
109 non’s theory of entropy (Shannon, 1948), with diversity being measured as
110 $H = -\sum_{i=1}^N p_i \log p_i$, where p_i is the proportion of the i -th pixel value (e.g.,
111 digital number, DN) found within a moving window containing N pixels.
112 Shannon’s H basically summarizes the partition of abundances (*sensu* Whit-
113 taker, 1965) by taking into account both relative abundance and richness of
114 DNs (Figure 1).

115 However, Shannon's entropy is a point descriptor of (remotely sensed)
116 diversity. As such, it shows only one part of the whole potential diversity
117 spectrum at a glance. The use of generalized entropies has been advocated to
118 face such problem. In this case, one single formula represents a parameterized
119 version of a diversity index, thus providing a continuum of potential diversity
120 indices. In the context of the measurement of diversity, the Rényi (1970)
121 parametric entropy

$$H_\alpha = \frac{1}{1-\alpha} \log \sum_{i=1}^N p_i^\alpha \quad (1)$$

122

123 with $0 \leq \alpha \leq \infty$ represents a powerful tool to account for the continuum
124 of diversity (Figure 1).

125 One particularly convenient property of H_α is that by varying the pa-
126 rameter α there is a continuum of possible diversity measures, which differ
127 in their sensitivity to rare and abundant DN, becoming increasingly dom-
128 inated by the most common DN for increasing values of α . Note that for
129 $\alpha \rightarrow 1$, H_1 equals the Shannon's entropy. A similar formulation was then
130 proposed by Hill (1973) who expressed parametric diversity as the "numbers
131 equivalent" of Rényi generalized entropy. Appendix S1 provides the original
132 formulation.

133 Rényi (and Hill) parametric functions summarize diversity by taking into
134 account the pixel values of a satellite image and their relative abundances.
135 However, they do not allow to explicitly consider the differences among these
136 values. As an example, two arrays of 9 pixels with maximum richness and
137 evenness (i.e. both containing 9 different DN with relative abundances
138 $p_i = \frac{1}{9}$) but differing in their values will attain the same Shannon diversity
139 irrespective of the values of the DN in both arrays.

140 By introducing a distance parameter d_{ij} among each pair of values i and
141 j , Rao's quadratic entropy (Rao , 1982)

$$Q = \sum_{i,j=1}^N p_i p_j d_{ij} \quad (2)$$

142

143 explicitly considers the differences among the pixel values in the calcula-
144 tion of diversity (Figure 1). Hence, two different pixels with values [2,3] will
145 attain a lower diversity with respect to two pixels with values [0,100]. For
146 instance, to make an ecological parallel, this is somewhat similar to the phy-
147 logenetic distance between two species: the values [2,3] would be equivalent

148 to two sister species closely related on the tree of life while [1,100] would be
149 equivalent to two very distant species on the tree of life.

150 The aim of this paper is thus to propose, for the first time, a parameter-
151 ization of Rao's quadratic entropy in order to provide a generalized entropy
152 which accounts for both relative abundances and distances among pixel val-
153 ues. The proposed approach is now part of the `rasterdiv` R package, a
154 package dedicated to diversity measures of spatial matrices, increasing its
155 capability to discern among different diversity measures by a single formula.

156 2 Spatio-parametric Rao's quadratic entropy

157 Inter-pixel spectral distances are directly related to landscape heterogeneity
158 and they are capable of describing species habitats, starting with a satellite
159 image (Rocchini et al., 2005). A satellite image can be viewed as a matrix of
160 numbers describing Earth reflectance in different dimensions stored as pixels.
161 A sensor per each light wavelength records the reflectance of a certain object
162 in that wavelength which are stored into numbers in a certain range (e.g.,
163 digital numbers in 8 bits, ranging from 0 to 255). In general, the higher the
164 variability in the spectral space defined by the pixel reflectance values, the
165 higher the diversity of the ecosystem under study.

166 Consider a window of N pixels moving across the whole image to cal-
167 culate a diversity index. Let i and j be two pixels randomly chosen with
168 repetition within the moving window. Let d_{ij} be a symmetric measure of
169 the (multi)spectral distance between i and j such that $d_{ij} = d_{ji}$ and $d_{ii} = 0$.
170 Rao's Q (Rao , 1982) is defined as:

$$Q = \sum_{i,j=1}^N p_i p_j d_{ij} = \sum_{i,j=1}^N \frac{1}{N} \times \frac{1}{N} d_{ij} \quad (3)$$

171
172 Therefore, Q measures the expected (i.e. mean) distance between two
173 randomly chosen pixels and $\frac{1}{N}$ is the probability to extract each pixel. Note
174 that, unlike H_α or K_α the calculation of Rao's quadratic entropy is not
175 limited to single bands but can be extended to multispectral systems of any
176 dimension. For the connection between quadratic entropy and variance, see
177 Rocchini et al., 2019.

178 Two parametric versions of quadratic entropy have been proposed by Ri-
179 cotta and Szeidl (2006) and Leinster and Cobbold (2012). These parametric
180 formulas were aimed at reconciling Rao's Q with parametric entropies. How-
181 ever, they have only been rarely used in practice.

182 A more direct approach for developing a parametric version of quadratic
 183 entropy stems from the work of Guiasu and Guiasu (2011). Let $\omega_{ij} = \frac{1}{N} \times \frac{1}{N}$
 184 be the combined probability of selecting pixels i and j in this order. Guiasu
 185 and Guiasu (2011) noted that Rao's Q can be expressed as a linear function
 186 of the combined probabilities of all pairs of pixels:

$$Q = \sum_{i,j=1}^N \omega_{ij} d_{ij} = \sum_{i,j=1}^N \frac{1}{N} \times \frac{1}{N} d_{ij} = \sum_{i,j=1}^N \frac{1}{N^2} d_{ij} \quad (4)$$

187

188 In practice, Rao's Q is the arithmetic mean of the distances d_{ij} between
 189 all pairs of pixels i and j . Hence, in order to implement a parametric version
 190 of Rao's Q , it seems natural to substitute the arithmetic mean in Equation
 191 4 with a generalized mean (Hardy et al., 1952):

$$Q_\alpha = \left(\sum_{i,j=1}^N \omega_{ij} d_{ij}^\alpha \right)^{\frac{1}{\alpha}} = \left(\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^\alpha \right)^{\frac{1}{\alpha}} \quad (5)$$

192

193 This operation connects Q_α with other diversity metrics that are ex-
 194 pressed as generalized means, such as Hill's (Hill, 1973) or Jost's (Jost ,
 195 2006) numbers (Appendix S1) equivalents (see also Leinster and Cobbold,
 196 2012).

197 The Rao's Q , viewed as an arithmetic mean, is one of all the possible
 198 means in its generalized form Q_α :

$$Q_\alpha = \begin{cases} \alpha \rightarrow 0, Q_0 = \sqrt[N^2]{\prod_{i,j=1}^N d_{ij}} & \leftarrow \text{geometric} \\ \alpha = 1, Q_1 = Q = \sum_{i,j=1}^N \frac{1}{N^2} d_{ij} & \leftarrow \text{arithmetic} \\ \alpha = 2, Q_2 = \sqrt{\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^2} & \leftarrow \text{quadratic} \\ \alpha = 3, Q_3 = \sqrt[3]{\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^3} & \leftarrow \text{cubic} \\ \alpha \rightarrow \infty, Q_{\alpha \rightarrow \infty} = \max d_{ij} & \leftarrow \text{max}_d \end{cases} \quad (6)$$

199

200 The mathematical proof that i) for $\alpha \rightarrow 0$ Q_0 corresponds to the geometric
 201 mean, and ii) for $\alpha \rightarrow \infty$ Q_∞ corresponds to the maximum distance between
 202 pixel values pairs is provided in Appendix S1.

203 Each generalized mean always lies between the smallest and largest of its
 204 values. Increasing the parameter α will increase the weight of the highest

205 values of d_{ij} , thus providing a continuum of potential diversity indices (Figure
206 1).

207 3 The algorithm

Starting from a satellite image, a spatial moving window might be used to make the calculation on predefined extents of analysis. The grain (*sensu* Dungan et al., 2002) will be the resolution of the image while the extent of analysis will be the size of the moving window (Figure 2). The calculation is based on a distance matrix of type:

$$M_d = \begin{pmatrix} d_{\lambda_1, \lambda_1} & d_{\lambda_1, \lambda_2} & d_{\lambda_1, \lambda_3} & \cdots & d_{\lambda_1, \lambda_n} \\ d_{\lambda_2, \lambda_1} & d_{\lambda_2, \lambda_2} & d_{\lambda_2, \lambda_3} & \cdots & d_{\lambda_2, \lambda_n} \\ d_{\lambda_3, \lambda_1} & d_{\lambda_3, \lambda_2} & d_{\lambda_3, \lambda_3} & \cdots & d_{\lambda_3, \lambda_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{\lambda_n, \lambda_1} & d_{\lambda_n, \lambda_2} & d_{\lambda_n, \lambda_3} & \cdots & d_{\lambda_n, \lambda_n} \end{pmatrix} \quad (7)$$

208

209 among all the potential pairs of pixels inside the moving window. The
210 diagonal terms of the matrix (which equal zero) will have no effect for $\alpha > 0$
211 (Equation 6), since they would enter the \sum term. On the contrary, for
212 $\alpha \rightarrow 0$, they would enter the \prod term by nullifying Q_0 .

213 We coded the proposed parameterization of Rao's quadratic entropy as an
214 R function, implementing the previously developed `rasterdiv` package (Mar-
215 cantonio et al. (2020), <https://CRAN.R-project.org/package=rasterdiv>).
216 The calculation of different Q_α by automatically changing the range of po-
217 tential α values is done by the function `paRao`, as:

```
218 > paRao(x, alpha=c(0:4, Inf), method="classic", 1  
219   dist_m="euclidean", window=9, na.tolerance=0.5, simplify=3,  
220   np=8, cluster.type="SOCK", diag=TRUE) 3
```

221 where `x` is the input dataset which can be a `RasterLayer` or a matrix class
222 object, `alpha` is the α parameter of Equation 5, which can be a single value
223 or a vector of integers. In the example above, α is a vector of integers ranging
224 from 0 to 4, plus `Inf`, which in the R language is a reserved word representing
225 positive infinity ($\alpha \rightarrow \infty$). The option `method` decides if `paRao` is calculated
226 with 1 single layer (`classic`) or with more than one layer (`multidimension`).
227 With `method="multidimension"` then `x` must be a list of objects. `dist_m`
228 is the type of distance considered in the calculation of the index, and can
229 be set to any distance class implemented in the R package `proxy`, such as
230 "euclidean", "canberra" or "manhattan". Moreover, `dist_m` can also be

231 an user-defined matrix of distances. However, if `method` is set to "classic"
232 (unidimensional paRao) all distance types reduce to the Euclidean distance.
233 The argument `window` is the side length in cells of the moving window (in
234 this case set to 9), whereas `na.tolerance` is the proportion (0-1) of NA's
235 cell allowed in a moving window: if the proportion of NA's cells in a moving
236 window exceeds `na.tolerance` then the value of the moving window central
237 pixel will be NA. The option `simplify` allows to reduce the number of
238 decimal places to ease the calculation by reducing the number of numerical
239 categories, i.e., if `simplify=3` only the first three digits of data will be con-
240 sidered for the calculation of the index. `np` is the number of parallel processes
241 used in the calculation. If `np>1` then the `doParallel` package will be called
242 for parallel calculation, and `cluster.type` will indicate the type of cluster
243 to be opened (default is "SOCK", "MPI" and "FORK" are the alternatives).
244 The `diag` argument refers to the diagonal term of Equation 7. It will have
245 no effect on the function for $\alpha > 0$, while it will nullify the value of Q_α if set
246 to TRUE, as previously explained in Equation 7.

247

248 3.1 Global test of the parametric Rao's Q variation 249 over the planet

250 We applied the algorithm to a Copernicus Proba-V NDVI (Normalized Dif-
251 ference Vegetation Index) long term average image (June 21st 1999-2017) at
252 5km grain, also provided in the `rasterdiv` package as a free `Rasterlayer`
253 dataset which can be loaded by the function `data()` (Figure 2). The para-
254 metric Rao algorithm can also be applied to multispectral data; in such a
255 case distances are calculated in the multisystem created by the values of the
256 pixels in each axis/band. The moving window passing throughout the whole
257 image will return M_{Q_α} matrices/layers where α is the value chosen in the R
258 function `paRao`.

259 With $\alpha \rightarrow 0$ the \prod in Equation 6 leads to zeroes throughout the whole
260 map (Figure 3). Increasing α will increase the weight of higher distances
261 among different values until reaching the maximum distance value for $\alpha \rightarrow$
262 ∞ . In this case the maximum turnover is reached and areas with maximum
263 β -diversity will be apparent. In this case, a multitemporal set is used (long
264 term average NDVI from June 21st 1999-2017). Hence, areas with the highest
265 spatial and temporal turnover are enhanced, namely major mountain ridges.
266 We expect that using single frame images would lead to the enhancement of
267 the spatial component of diversity.

268 Since the whole process is based on distances in a spectral space between

269 pairs of pixels in terms of their “spectral characters” or in the “spectral
270 space”, it is important to notice some cornerstone aspects on the use of
271 distances from satellite images, especially when comparing different images
272 or the same image in different times. In satellite images, the measure of
273 distances could be impacted by: ii) the use of different sensors with different
274 radiometric resolutions, as an example an 8-bit ($2^8 = 256$ values) with respect
275 to a 16-bit ($2^{16} = 65536$ values) image, or ii) the radiometric calibration
276 which has been performed, e.g. with a non-linear transform. Therefore,
277 care should be taken when making use of distances in remote sensing data,
278 explicitly taking into account how the vector of proportions between pixels
279 belonging to some defined classes (e.g., digital numbers, DN_s) was obtained.

280 The complete code of the function can be directly seen in R by typing the
281 `paRao` function name. Moreover, a complete R coding session, to perform
282 the above described analysis is provided in Appendix S2.

283 **3.2 Local case study: the diversity of vegetation green-** 284 **ness and the ecoregions of California**

285 A comparison between in-situ and remotely sensed diversity at worldwide
286 scale might be difficult due to known biases in e.g. sampling effort, tax-
287 onomies, spatial uncertainty (Rocchini et al., 2017). Hence, we decided to
288 calculate the Rao’s Q index on a NDVI raster layer of California (USA) to
289 be compared with data in the field on native plant species diversity provided
290 in Thornhill et al. (2017) from Baldwin et al. (2017). We chose California as
291 a case study due to its high ecological diversity as well as to the availability
292 of plant species field-data for this region.

293 In practice, we aimed at visualizing and describing differences in both
294 diversity and structure of vegetation for the state of California, USA. First,
295 an NDVI raster layer was derived from Copernicus Sentinel-2 data (European
296 Space Agency, reference period: January 2017 to July 2018) and processed
297 through Google Earth Engine to filter out cloud cover, select the greenest
298 pixel of the time series and resample at 100 m pixel resolution. Then, the
299 `paRao` R function was used to derive Rao’s Q index, considering both the
300 original formulation of the Rao’s Q ($\alpha = 1$, Equation 6) and the formulation
301 with $\alpha \rightarrow \infty$ maximizing β -diversity (Figure 3), with a moving window of
302 9x9 pixels.

303 A map of plant species richness was derived using the potential distribu-
304 tion range of 5,222 native California vascular plants modelled by Thornhill
305 et al. (2017). Moreover, a vector map reporting the ecoregions of California
306 (level III) was downloaded from the United States Environmental Protection

307 Agency. In Figure 4, we showed NDVI, the Rao's Q indices with $\alpha = 1$
308 and $\alpha \rightarrow \infty$ and plant species richness, reporting the boundaries of the dif-
309 ferent ecoregions for California. This comparison revealed macro-ecological
310 and bio-geographical patterns which can be better interpreted considering
311 the information condensed in the Rao's Q index.

312 For example, the ecoregion "Coast range" (labelled with 1 in Figure 4)
313 is composed by low mountains covered by highly productive, rain-drenched
314 evergreen forests. As a result, this region showed very high NDVI values
315 but a low Rao's Q index (low vegetation structural diversity) and low to
316 medium plant species richness. The adjacent "Klamath Mountain" ecoregion
317 (2) is instead characterized by highly dissected ridges, foothills, and valleys.
318 This region still showed high NDVI values but higher Rao's values with
319 respect to region 1, which resulted in a high plant species richness. The
320 diverse flora of this region, a mosaic of both northern Californian and Pacific
321 Northwestern conifers and hardwoods, is rich in endemic and relic species. A
322 similar pattern, although caused by opposite factors, was recognizable for the
323 "Central Valley" region of California (3), which is composed of flat, urbanized
324 and intensively farmed plains. The extensive presence of irrigated crops
325 intersected with urbanized areas caused medium to high NDVI values and
326 a very high apparent structural diversity. However, the same factors caused
327 a low native species richness, especially in the drier southern portion of the
328 valley. Finally, very dry and warm broad basins and scattered mountains
329 characterize the "Mohave and Sonora ranges" ecoregions (4) which showed
330 very low NDVI and Rao's Q values (with scattered higher values associated
331 with local topographical variability) and low native plant species richness.

332 Passing from the pure Rao's Q index ($\alpha=1$) to its parameterization with
333 $\alpha \rightarrow \infty$ helped to increase the discrimination among areas, due to the fact
334 that when $\alpha \rightarrow \infty$ the Rao's Q corresponds to the maximum distance (β -
335 diversity) among pixel values in a site. Very similar gradients of the spatial
336 heterogeneity of California (including BIOMOD variables, NDVI, elevation)
337 as well as environmental DNA (eDNA) data are found in Lin et al. (2020).

338 4 Discussion

339 In this paper, we provided a straightforward solution to: i) account for dis-
340 tances in an Information Theory based metric, and ii) provide a generalized
341 formula in order to avoid point description and account for the continuum of
342 diversity. Diversity can be represented by different dimensions (Nakamura
343 et al., 2020). Considering one single metric to account for the whole contin-
344 uum of diversity metrics might be a powerful addition to the main framework.

345 On the contrary, fragmenting the concept of diversity when trying to capture
346 single aspects of the whole spectrum could be counterproductive.

347 The proposed unifying measure succeeded to integrate abundance- and
348 distance-based algorithms over a wide variety of diversity metrics. We demon-
349 strated that such integration is not only theoretical but also applicable to
350 real spatial data, considering several dimensions of diversity at the same time.
351 Being part of the `rasterdiv` R package, the proposed method is expected to
352 ensure high robustness and reproducibility.

353 Remote sensing is obviously not a panacea for all the organismic based
354 diversities like taxonomic-, functional-, genetic-diversity but it can represent
355 an important exploratory tool to detect diversity hotspots and their changes
356 in space and time at the ecosystem level. First of all, it measures heterogene-
357 ity of the environment with indirect links to the biodiversity of both plant
358 and animal taxa, but also with potential discrepancies with species diversity,
359 as in the presented case study of the native plant species diversity of Cali-
360 fornia. This said, depending on the complexity and the resolution at which
361 the proposed parameterized Rao's Q is applied, it might allow finding new
362 insights on the ecological processes acting in a certain ecosystem to shape its
363 diversity. In this paper, the examples provided were based on a single NDVI
364 layer since i) it is a valuable index of vegetation health and ii) it is freely
365 available in the `rasterdiv` package to reproduce the code proposed in this
366 paper. We are aware that NDVI has very limited capacity to track diversity
367 in some habitats like dense forests, because it is saturated at dense vegeta-
368 tion. From this point of view, imaging spectroscopy offers higher informa-
369 tion content, also enabling plant functional trait retrievals (Jetz et al., 2016;
370 Schneider et al., 2019) as well as structural traits by LiDAR data (Schneider
371 et al., 2020). The application of the proposed algorithm to future spaceborne
372 imaging spectroscopy is promising. In other words, the algorithm has been
373 thought to be used with multiple layers, like a whole multispectral image or
374 the most meaningful Principal Components (Peres-Neto et al., 2005), or land
375 use classes probabilities derived from fuzzy set theory (Rocchini and Ricotta,
376 2007; Feoli, 2018). This is even one of the major advantages of the Rao's Q
377 metric which allows considering both abundance and distance among pixel
378 values, thus being applicable to any continuous raster layer, or to any matrix
379 combination, even in a multiple spectral system.

380 Creating a unique "umbrella" under which all of the potential metrics of
381 diversity can be used is highly beneficial for e.g. monitoring the variation in
382 time of biological systems considering two major axes: i) the α parameter in
383 Equation 5 providing information about the type of diversity at time t_0 , ii)
384 the temporal dimension from time t_0 to time t_n given the same α parameter.
385 For the future, exploring such temporal dimension would allow gathering

386 information of ecosystem changes in different diversity types at a glance.

387 Moreover, generalized entropy allows us to characterize the dimensionality
388 of diversity (*sensu* Stevens and Tello, 2014) of different habitats/ecosystems.
389 Those areas with a higher diversity dimensionality, namely a higher variabil-
390 ity into the diversity spectrum would need a generalized measure to be fully
391 undertaken. On the contrary, ecosystems with a lower dimensionality would
392 have a lower difference among the different diversity measures with a flat
393 curve of the diversity spectrum (Nakamura et al., 2020).

394 From a functional point of view, when all indices of diversity are highly
395 correlated to each other (low dimensionality), it is expected that the eco-
396 logical processes underlying diversity are just a few. On the contrary, with
397 a lower correlation among indices (higher dimensionality) there might be a
398 higher number of axes of variation coming out from different processes shap-
399 ing ecological heterogeneity in space (Stevens and Tello, 2014).

400 There might be the possibility that a completely random matrix produces
401 a pattern of diversity (Type I error). On the other side, a structured matrix
402 could produce a very low diversity pattern (Type II error, Gotelli (2000)). In
403 both cases, the parametric Rao's Q could allow to determine, thanks to the
404 use of a continuum of diversities, i) why a diversity pattern is still produced
405 even in case of a random matrix, and ii) why a certain landscape shows a very
406 low diversity in a certain point of the whole diversity spectrum. With point
407 descriptors of diversity such inference cannot be done since the investigation
408 is limited to a small window of the entire diversity spectrum, by basically
409 relying on a single final number. In other words, the commonly asked ques-
410 tion about what is the index which best describes diversity has no certain
411 answer (Gorelick, 2011). Hence, the use of a trend of diversities will lead to
412 the comprehension of hidden parts of the whole diversity dimensionality.

413 Furthermore, it is expected that the ecological processes shaping diversity
414 should act at defined spatial scales (Borcard and Legendre, 2002). Hence,
415 different diversity types of the whole dimensionality spectrum are expected
416 to show scale dependent patterns, being apparent only at certain scales and
417 not at some others. The use of a continuum allows measuring the different
418 diversity types altogether in a single step. Changing the extent of analysis
419 by different moving windows would then allow to encompass different spatial
420 structures at different scales.

421 While geographic gradients of diversity over space are complex to catch in
422 their very nature, biodiversity measurement has mainly relied in the past on
423 few formulas which represented an hegemony (Stevens et al., 2013). In this
424 paper, we demonstrated that diversity is actually multifaceted and should be
425 necessarily approached through a generalized approach.

426 5 Conclusion

427 In order to unfold the dimensionality of diversity methods to directly account
428 for several aspects of diversity at a time are needed. From this point of view,
429 generalized entropy undoubtedly represents a powerful approach for mapping
430 the diversity continuum.

431 Furthermore, it might be profitably used to plot multitemporal trends
432 (see e.g. Dornelas et al., 2014) of diversity metrics and discover previously
433 imperceptible differences when making use of single metrics (Figure 5).

434 Metrics grounded in Information Theory ensure to make use of relative
435 abundance of pixel values given the same richness in the moving window of
436 analysis. However, distance metrics allow to also account for the relative
437 dispersion in the spectral space of the cloud of pixels in a certain area (Lal-
438 iberté et al., 2020). The proposed parameterization of the Rao's Q explicitly
439 considers the dispersion of pixel values in a spectral space (and their relative
440 abundance) by allowing catching the whole dimensionality of diversity.

441 6 Data availability

442 The code and the data used in this paper are based on completely Free
443 and Open Source Software, and they are available at the CRAN reposi-
444 tory of the R package rasterdiv: [https://CRAN.R-project.org/package=](https://CRAN.R-project.org/package=rasterdiv)
445 `rasterdiv`.

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453 References

454 Baldwin, B.G., Thornhill, A.H., Freyman, W.A., Ackerly, D.D., Kling, M.M.,
455 Morueta-Holme, N., Mishler, B.D. (2017). Species richness and endemism
456 in the native flora of California. *American Journal of Botany*, 104: 1-15.

- 457 Borcard, D., Legendre, P. (2002). All-scale spatial analysis of ecological data
458 by means of principal coordinates of neighbour matrices. *Ecological Mod-*
459 *elling*, 153: 51-68.
- 460 Dornelas, M., Gotelli, N.J., McGill, B., Shimadzu, H., Moyes, F., Sievers, C.,
461 Magurran, A.E. (2014). Assemblage time series reveal biodiversity change
462 but not systematic loss. *Science*, 344: 296-299.
- 463 Evans, M.R., Grimm, V., Johst, K., Knuuttila, T., de Langhe, R., Lessells,
464 C.M., Merz, M., O'Malley, M.A., Orzack, S.H., Weisberg, M., Wilkinson,
465 D.J., Wolkenhauer, O., Benton, T.G. (2013). Do simple models lead to
466 generality in ecology? *Trends in Ecology & Evolution*, 28: 578-583.
- 467 Ferrier, S., Manion, G., Elith, J., Richardson, K. (2007). Using generalized
468 dissimilarity modelling to analyse and predict patterns of beta diversity in
469 regional biodiversity assessment. *Diversity and Distributions*, 13: 252-264.
- 470 Dungan, J.L., Perry, J.N., Dale, M.R.T., Legendre, P., Citron-Pousty, S.,
471 Fortin, M.-J., Jakomulska, A., Miriti, M. and Rosenberg, M.S. (2002). A
472 balanced view of scale in spatial statistical analysis. *Ecography*, 25: 626-
473 640.
- 474 Gillespie, T.W. (2005). Predicting woody-plant species richness in tropical
475 dry forests: a case study from South Florida, USA. *Ecological Applications*,
476 15: 27-37.
- 477 Gorelick, R. (2011). Do we have a consistent terminology for species diver-
478 sity? The fallacy of true diversity. *Oecologia*, 167: 885-888.
- 479 Null model analysis of species co-occurrence patterns. *Ecology*, 81: 2606-
480 2621.
- 481 Goodall, D.W. (1970). Statistical ecology, p. 99-124. In Johnston, R.F., ed.
482 Annual review of ecology and systematics, Vol. 1. Annual Reviews, Palo
483 Alto, California, USA.
- 484 Guiasu, R.C., Guiasu, S. (2011). The weighted quadratic index of biodiversity
485 for pairs of species: a generalization of Rao's index. *Natural Science*, 3:
486 795-801.
- 487 Hardy, G., Littlewood, J.E., Polya, G. (1952). *Inequalities*. Cambridge Uni-
488 *versity Press*, Cambridge, UK.
- 489 Hill, M.O. (1973). Diversity and evenness: a unifying notation and its con-
490 *sequences*. *Ecology*, 54: 427-431.

- 491 Hobohm, C., Janisova, M., Steinbauer, M., Landi, S., Field, R., Vander-
492 plank, S., Beierkuhnlein, C., Grytnes, J.-A., Vetaas, R.O., Fidelis, A., de
493 Nascimento, L., Clark, V.P., Fernandez-Palacios, J.M., Franklin, S., Guar-
494 ino, R., Huang, J., Krestov, P., Ma, K., Onipchenko, V., Palmer, M.W.,
495 Fragomeni Simon, M., Stolz, C., Chiarucci, A. (2019). Global endemics-
496 area relationships of vascular plants. *Perspectives in Ecology and Conser-
497 vation*, 17: 41-49.
- 498 Hutchinson, G. 1959. Homage to Santa Rosalia or why are there so many
499 kinds of animals? *American Naturalist*, 93: 145-159.
- 500 Feoli, E. (2018). Classification of plant communities and fuzzy diversity of
501 vegetation systems. *Community Ecology*, 19: 186-198.
- 502 Jetz, W., Cavender-Bares, J., Pavlick, R., Schimel, D., Davis, F.W., Asner,
503 G.P., Guralnick, R., Kattge, J., Latimer, A.M., Moorcroft, P., Schaep-
504 man, M.E., Schildhauer, M.P., Schneider, F.D., Schrod, F., Stahl, U.,
505 Ustin, S.L. (2016). Monitoring plant functional diversity from space. *Nature
506 Plants*, 2: 16024.
- 507 Johnson, P.C.D., Barry, S.J.E., Ferguson, H.M., Müller, P. (2015). Power
508 analysis for generalized linear mixed models in ecology and evolution.
509 *Methods in Ecology and Evolution*, 6: 133-142.
- 510 Jost, L. (2006). Entropy and diversity. *Oikos*, 113: 363-375.
- 511 Laliberté, E., Schweiger, A.K., Legendre, P. (2019). Partitioning plant spec-
512 tral diversity into alpha and beta components. *Ecology Letters*, 23: 370-
513 380.
- 514 Leinster, T., Cobbold, C.A. (2012). Measuring diversity: the importance of
515 species similarity. *Ecology*, 93: 477-489.
- 516 Leitão, P.J., Schwieder, M., Suess, S., Catry, I., Milton, E.J., Moreira, F.,
517 Osborne, P.E., Pinto, M.J., van der Linden, S., Hostert, P. (2015), Map-
518 ping beta diversity from space: Sparse Generalised Dissimilarity Modelling
519 (SGDM) for analysing high-dimensional data. *Methods in Ecology and
520 Evolution*, 6: 764-771.
- 521 Lin, M., Levi Simons, A., Curd, E.E., Harrigan, R.J., Schneider, F.D.,
522 Ruiz-Ramos, D.V., Gold, Z., Osborne, M.G., Shirazi, S., Schweizer, T.M.,
523 Moore, T.N., Fox, E.A., Turba, R., Garcia-Vedrenne, A.E., Helman, S.K.,
524 Rutledge, K., Palacios Mejia, M., Munguia Ramos, M.N., Wetzler, R.,
525 Pentcheff, D., McTavish, E.J., Dawson, M.N., Shapiro, B., Wayne, R.K.,

- 526 Meyer, R.S. (2020). A biodiversity composition map of California derived
527 from environmental DNA metabarcoding and Earth observation. bioRxiv
528 2020.06.19.160374. doi: <https://doi.org/10.1101/2020.06.19.160374>
- 529 MacArthur, R.H., Recher, H., Cody, M. (1966). On the relation between
530 habitat selection and species diversity. *American Naturalist*, 100: 319-327.
- 531 Marcantonio, M., Iannacito, M., Thouverai, E., Da Re, D., Tattoni, C.,
532 Bacaro, G., Vicario, S., Rocchini, D. (2020). rasterdiv: Diversity Indices for
533 Numerical Matrices. R package version 0.2-0. <https://CRAN.R-project.org/package=rasterdiv>
534
- 535 Nakamura, G., Gonçalves, L.O., Duarte, L.d.S. (2020). Revisiting the dimen-
536 sionality of biological diversity. *Ecography*, 43: 539-548.
- 537 Patil, G.P., Taillie, C. (1982). Diversity as a concept and its measurement.
538 *Journal of the American Statistical Association*, 77: 548-561.
- 539 Peres-Neto, P.R., Jackson, D.A., Somers, K.M. (2005). How many principal
540 components? stopping rules for determining the number of non-trivial axes
541 revisited. *Computational Statistics & Data Analysis*, 49: 974-997.
- 542 Rao, C.R. (1982). Diversity and dissimilarity coefficients: a unified approach.
543 *Theoretical Population Biology*, 21: 24-43.
- 544 Rényi, A., 1970. *Probability Theory*. North Holland Publishing Company,
545 Amsterdam.
- 546 Ricotta, C., Szeidl, L. (2006). Towards a unifying approach to diversity mea-
547 sures: bridging the gap between the Shannon entropy and Rao's quadratic
548 index. *Theoretical Population Biology*, 70: 237-243.
- 549 Rocchini, D., Andreini Butini, S., Chiarucci, A. (2005). Maximizing plant
550 species inventory efficiency by means of remotely sensed spectral distances.
551 *Global Ecology and Biogeography*, 14: 431-437.
- 552 Rocchini, D., Delucchi, L., Bacaro, G., Cavallini, P., Feilhauer, H., Foody,
553 G.M., He, K.S., Nagendra, H., Porta, C., Ricotta, C., Schmidtlein, S.,
554 Spano, L.D., Wegmann, M., Neteler, M. (2013). Calculating landscape
555 diversity with information-theory based indices: A GRASS GIS solution.
556 *Ecological Informatics*, 17: 82-93.
- 557 Rocchini, D., Garzon-Lopez, C.X., Marcantonio, M., Amici, V., Bacaro, G.,
558 Bastin, L., Brummitt, N., Chiarucci, A., Foody, G.M., Hauffe, H.C., He,

- 559 K.S., Ricotta, C., Rizzoli, A., Rosá, R. (2017). Anticipating species dis-
560 tributions: handling sampling effort bias under a Bayesian framework.
561 *Science of the Total Environment*, 584-585, 282-290.
- 562 Rocchini, D., Hortal, J., Lengyel, S., Lobo, J.M., Jiménez-Valverde, A., Ri-
563 cotta, C., Bacaro, G., Chiarucci, A. (2011). Accounting for uncertainty
564 when mapping species distributions: The need for maps of ignorance.
565 *Progress in Physical Geography*, 35: 211-226.
- 566 Rocchini, D., Luque, S., Pettorelli, N., Bastin, L., Doktor, D., Faedi, N.,
567 Feilhauer, H., Féret, J.-B., Foody, G.M., Gavish, Y., Godinho, S., Kunin,
568 W.E., Lausch, A., Leitão, P.J., Marcantonio, M., Neteler, M., Ricotta,
569 C., Schmidtlein, S., Vihervaara, P., Wegmann, M., Nagendra, H. (2018).
570 Measuring β -diversity by remote sensing: a challenge for biodiversity mon-
571 itoring. *Methods in Ecology and Evolution*, 9: 1787-1798.
- 572 Rocchini, D., Marcantonio, M., Da Re, D., Chirici, G., Galluzzi, M., Lenoir,
573 J., Ricotta, C., Torresani, M., Ziv, G. (2019). Time-lapsing biodiversity:
574 an open source method for measuring diversity changes by remote sensing.
575 *Remote Sensing of Environment*, 231: 111192.
- 576 Rocchini, D., Ricotta, C. (2007). Are landscapes as crisp as we may think?
577 *Ecological Modelling*, 204: 535-539.
- 578 Schimel, D., Schneider, F.D., JPL Carbon and Ecosystem Participants
579 (2019). Flux towers in the sky: global ecology from space. *New Phytol-
580 ogist*, 224: 570-584.
- 581 Schneider, F.D., Ferraz, A., Schimel, D. (2019). Watching Earth's Intercon-
582 nected Systems at Work. *Eos*, 100.
- 583 Schneider, F.D., Ferraz, A., Hancock, S., Duncanson, L.I., Dubayah, R.O.,
584 Pavlick, R.P., Schimel, D.S. (2020). Towards mapping the diversity of
585 canopy structure from space with GEDI. *Environmental Research Letters*,
586 15, 115006.
- 587 Shannon, C.E. (1948). A mathematical theory of communication. *Bell System
588 Technical Journal*, 27: 379-423, 623-656.
- 589 Stevens, R.D., Tello, J.S. (2014). On the measurement of dimensionality of
590 biodiversity. *Global Ecology and Biogeography*, 23: 1115-1125.
- 591 Stevens, R.D., Tello, J.S., Gavilanez, M.M. (2013). Stronger tests of mech-
592 anisms underlying geographic gradients of biodiversity: insights from the
593 dimensionality of biodiversity. *PLOS ONE*, 8: e56853.

- 594 Thornhill, A.H., Baldwin, B.G., Freyman, W.A., Nosratinia, S., Kling, M.M.,
595 Morueta-Holme, N., Madsen, T.P., Ackerly, D.D., Mishler, B.D. (2017).
596 Spatial phylogenetics of the native California flora. *BMC Biology* 15: 96.
- 597 Whittaker, R.H. (1965). Dominance and diversity in land plant communities.
598 *Science*, 147: 250-260.
- 599 Zellweger, F., De Frenne, P., Lenoir, J., Rocchini, D., Coomes, D. (2019).
600 Advances in microclimate ecology arising from remote sensing. *Trends in*
601 *Ecology & Evolution*, 34: 327-341.

602 **Figures**

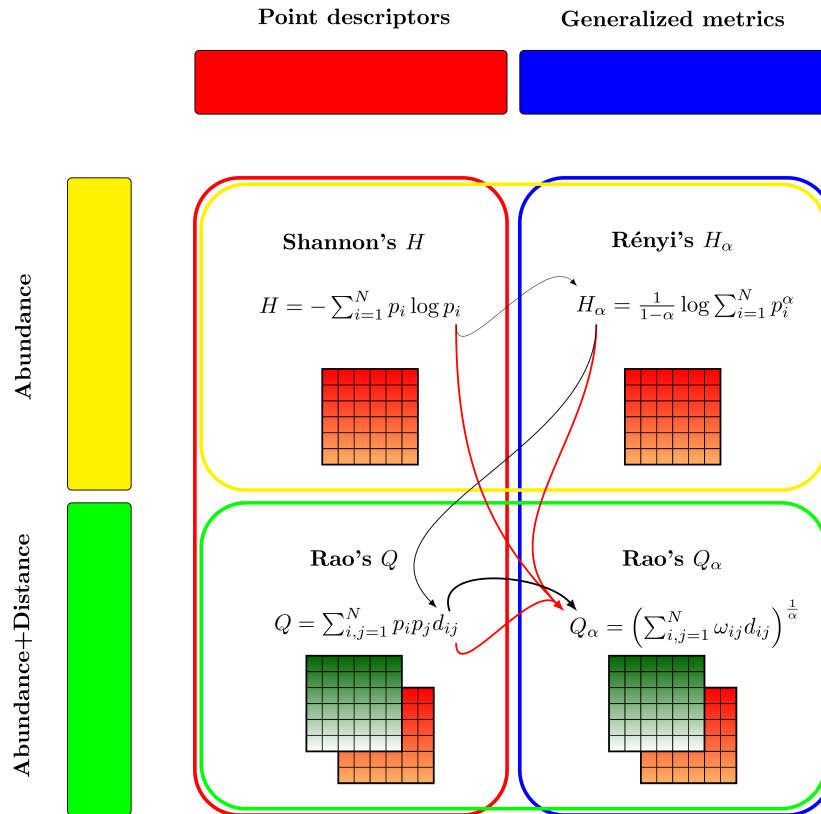


Figure 1: Grounding theory of this paper. Diversity measures can encompass abundance-based as well as abundance-distance-based metrics (yellow and green boxes, respectively). Abundance-distance-based metrics allow multiple layers to be used. The black lines represent the theoretical flow of this paper, with the thickness representing the complexity of each index, starting from Shannon's Information Theory (point descriptor) to Rényi's H_α (generalized entropy), which do not make use of distance. Distance enters the Rao's Q formula, but this is still a point descriptor of diversity. Finally, parametric Rao's Q_α comprises the use of distances and the generalized entropy concept. The red arrows represent the properties of the Rao's Q_α : i) it is grounded in Information Theory starting from Shannon's H , ii) it is a generalized entropy like the Rényi H_α , and iii) it makes use of distances like the Rao's Q .

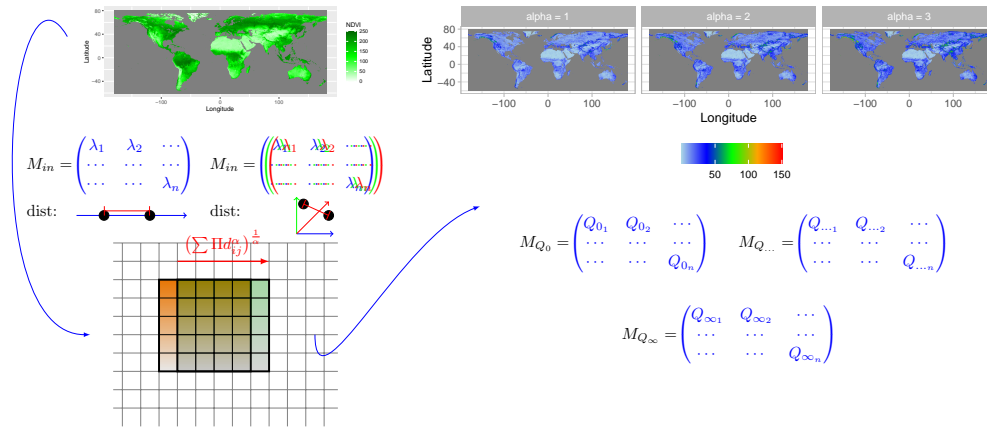


Figure 2: Starting from Copernicus Proba-V NDVI (Normalized Difference Vegetation Index) long term average image (June 21st 1999-2017) at 5km grain, parametric Rao's Q is calculated in a moving window. In this paper NDVI was used as a single layer to calculate distances on one axis, but several layers can be used as well. In this example, three layers (blue, green and red matrices) are shown to calculate distances. The algorithm is based on a moving window passing throughout the whole image, calculating the Rao's Q_{α} and saving the output in the central pixel. In this example a moving window of 5x5 pixels is passing (red arrow) from one position (orange) to the other (green). The output is a stack of layers each of which represents a different mean of the whole generalized mean spectrum of Equation 5.

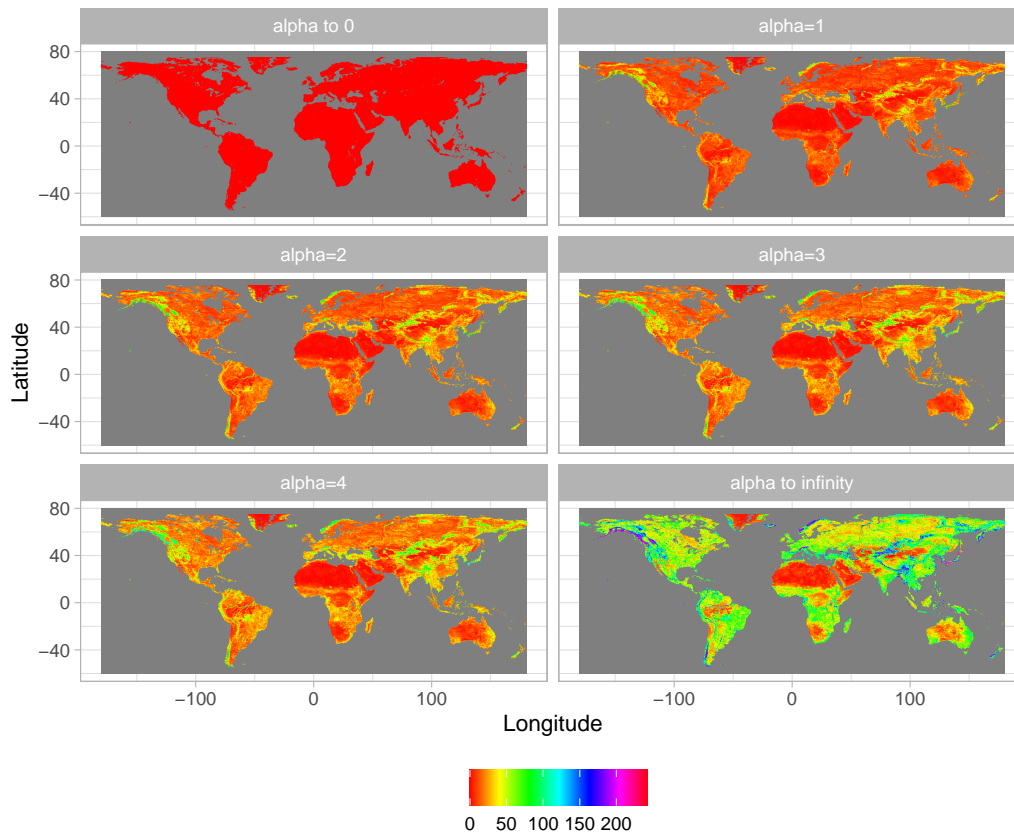


Figure 3: Output of the application of the algorithm shown in Figure 2, achieved by applying different α values: from 0 to 4 until $\alpha \rightarrow \infty$. The higher the value of the parameter α , the higher the weight of highest distances among pixel values, until reaching the maximum potential β -diversity (maximum distance) at $\alpha \rightarrow \infty$.

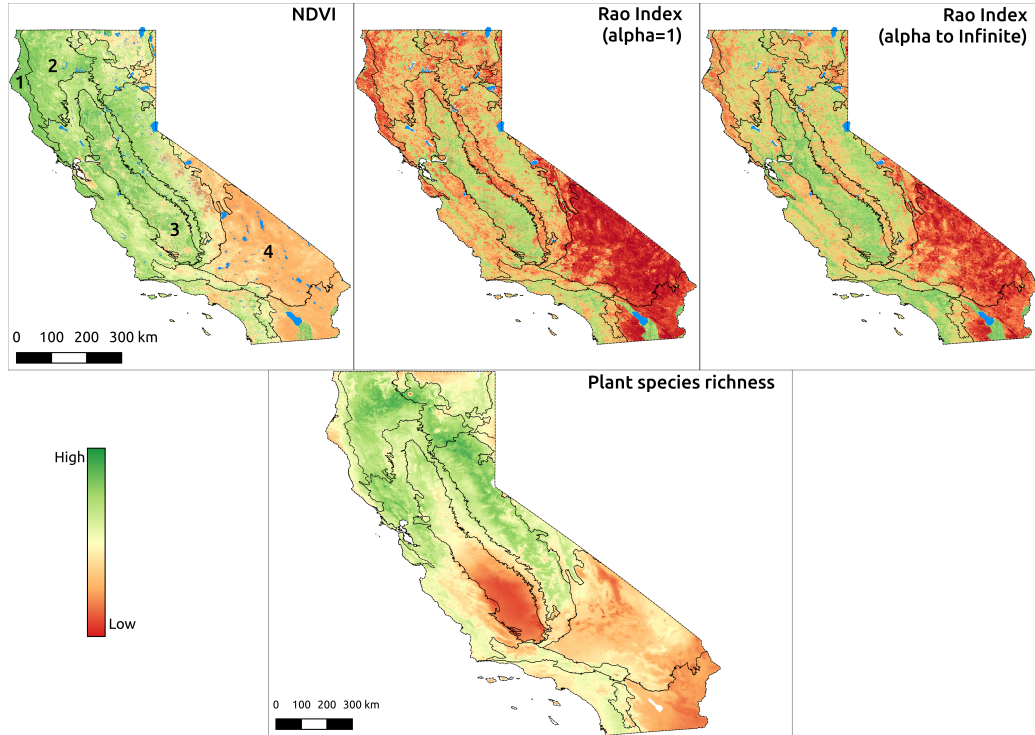


Figure 4: Comparison between NDVI, Rao's Q Index, native plant species richness for the ecoregions of California. The NDVI values shown in the top-left box (100 m resolution) were derived from the ESA Copernicus Sentinel-2 dataset then processed with Google Earth Engine and range between -0.26 (red) and 0.99 (green). The Rao's Q index shown in the top-right box was calculated from the NDVI map with $\alpha=1$ and α to infinite and a moving window of 9x9 pixels. High values are shown in dark green and represent pixel whose surrounding NDVI values are more "diverse" than pixel reported in red. The map reporting the potential native plant species richness of California (resolution: 810 m) was derived summing the binary potential distribution range of 5,222 native plant species modelled by Thornhill et al. (2017) and ranges between 134 (red) to 1029 (green) species per pixel (1 km^2). The ecoregions considered in this paper are overlapped to the NDVI image: 1) Coast range (low mountains covered by highly productive, rain-drenched evergreen forests), 2) Klamath Mountain (highly dissected ridges, foothills, and valleys), 3) Central Valley (flat, urbanized and intensively farmed plains), 4) Mohave and Sonora ranges (very dry and warm broad basins).

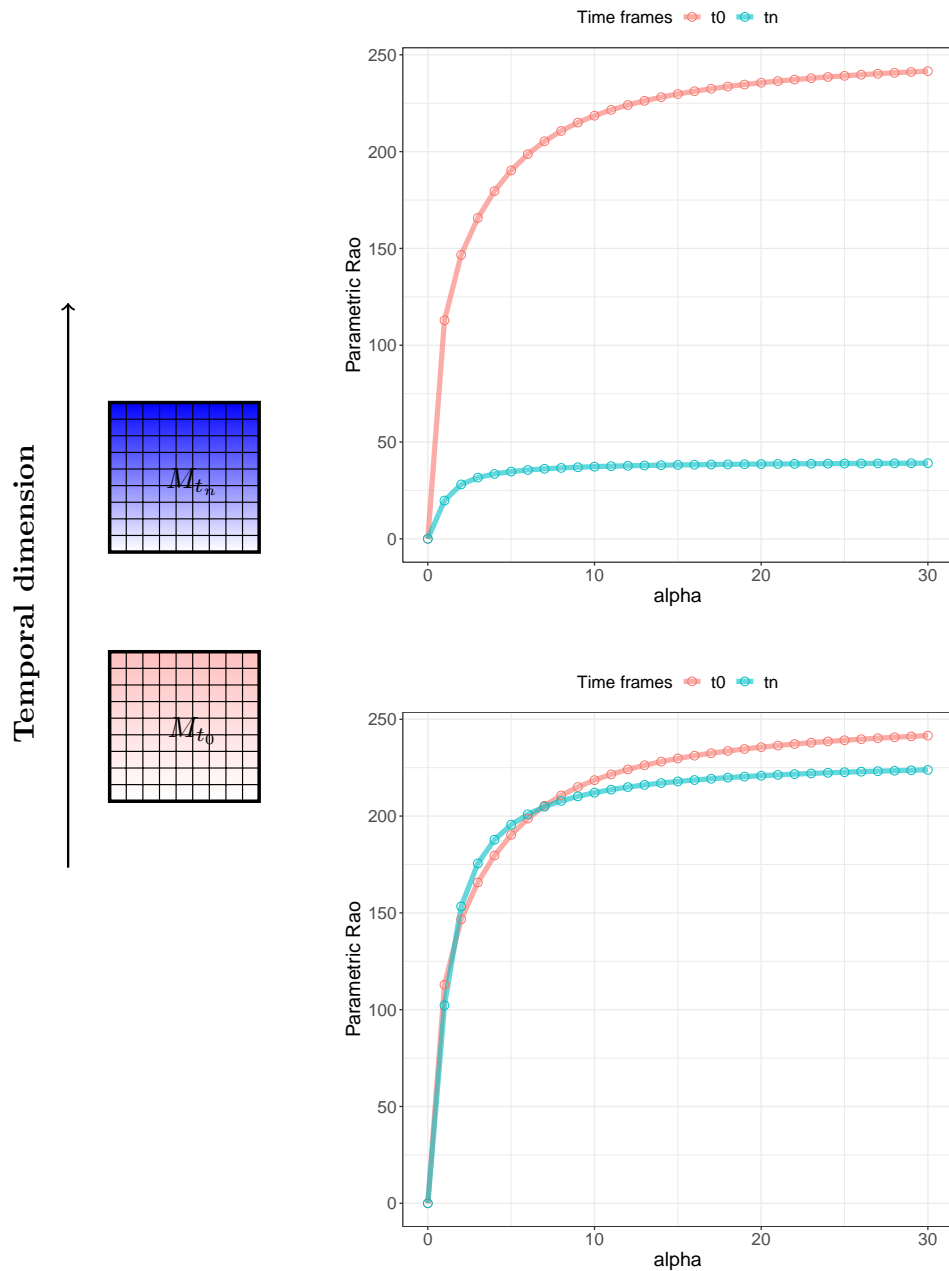


Figure 5: A theoretical example of the power of using generalized entropy for monitoring purposes. Given a landscape at times t_0 (pink) and t_n (blue), calculating generalized entropy will allow the formation of a graph showing the continuum of Rao's Q values observed over a range of values for α . The same landscape in different times might show an abrupt change (e.g., a catastrophic event) with an apparent diversity decrease (top). In this case, point descriptors (e.g., single α values) of diversity may be sufficient to describe this pattern. When the change in diversity is subtle (bottom), using a point descriptor might fail to detect it but it becomes manifest in the continuum of diversities based on generalized entropy. The complete code for reproducing this theoretical example is available in Appendix S3.

1 Appendix S1 - Mathematical dissertation on
2 the proposed algorithms
3 From zero to infinity: minimum to maximum
4 diversity of the planet by spatio-parametric
5 Rao's quadratic entropy

6

7 January 23, 2021

8 1 Hill's numbers and generalized entropy

9 Hill (1973) expressed parametric diversity as the “numbers equivalent” of
10 Rényi's generalized entropy, as:

$$K_\alpha = \frac{1}{\left(\sum_{i=1}^N p_i \times p_i^{\alpha-1}\right)^{\frac{1}{\alpha-1}}} \quad (1)$$

11
12 where the numbers equivalent K_α is the theoretical number of equally-
13 abundant DN's (i.e. all those with $p_i = \frac{1}{K_\alpha}$) that are needed in order that its
14 diversity be H_α (?).

15 Hill's K_α has the form of the reciprocal of a generalized mean of order $\alpha -$
16 1. Jost (2006) further showed that, like for H_α , the numbers equivalents of all
17 parametric and non-parametric measures of diversity that can be expressed as
18 monotonic functions of $\sum p_i^\alpha$ have the form of the reciprocal of a generalized
19 mean of order $\alpha - 1$ (for details, Jost , 2006).

20 2 Mathematical proof: for $\alpha \rightarrow 0$ Q_0 is the ge- 21 ometric mean among the generalized means, 22 for $\alpha \rightarrow \infty$ Q_∞ is the maximum distance be- 23 tween pixel values pairs

We want to compute

$$\lim_{\alpha \rightarrow 0} Q_\alpha \quad \text{where} \quad Q_\alpha = \left(\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^\alpha\right)^{\frac{1}{\alpha}}. \quad (2)$$

By $\exp(\log(x)) = x$ we can rewrite Q_α as

$$Q_\alpha = \left(\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^\alpha\right)^{\frac{1}{\alpha}} = \exp\left(\log\left(\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^\alpha\right)^{\frac{1}{\alpha}}\right) = \exp\left(\frac{1}{\alpha} \log\left(\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^\alpha\right)\right)$$

24 reminding that if $N > 1$, there is at least one distance $d_{ij} > 0$. We use this
25 last expression to calculate (2). We use the following two well known results.

26 **Theorem 1** (De l'Hôpital). *Let $f_1, g_1 : (a, b) \mapsto \mathbb{R}$ be two functions such that*

- 27 • $\lim_{x \rightarrow a} f_1(x) = \lim_{x \rightarrow a} g_1(x) = 0$

28 • f_1 and g_1 are differentiable in (a, b) with $g_1'(x) \neq 0$ for every $x \in (a, b)$

29 • the limit $\lim_{x \rightarrow a} \frac{f_1'(x)}{g_1'(x)} = L$ with $L \in \mathbb{R}$

then

$$\lim_{x \rightarrow a} \frac{f_1(x)}{g_1(x)} = L.$$

30 **Theorem 2** (Limit composition). Let $f_2 : (a, b) \mapsto \mathbb{R}$ and let $g_2 : (c, d) \mapsto \mathbb{R}$
 31 be two functions such that the image set of g_2 is contained in the domain of
 32 f_2 , i.e. $\text{Img}(g_2) \subseteq (a, b)$. Let $x_0 \in (c, d)$, if it holds that

33 • $\lim_{x \rightarrow x_0} g_2(x) = y_0$ with $g_2(x) \neq y_0$ definitely for $x \rightarrow x_0$

34 • $\lim_{y \rightarrow y_0} f_2(y) = l$

with $a, b, c, d, x_0, y_0, l \in \mathbb{R} \cup \pm\infty$ then

$$\lim_{x \rightarrow x_0} (f_2 \circ g_2)(x) = l.$$

We apply Theorem (2) to calculate the limit (2) with $f_2(x) = \exp(x)$ and

$$g_2(\alpha) = \frac{1}{\alpha} \log \left(\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^\alpha \right).$$

(all assumptions of the theorem hold). Setting $x_0 = 0$, we have to compute

$$\lim_{\alpha \rightarrow 0} g_2(\alpha). \quad (3)$$

which will be accomplished using Theorem (1) by setting $f_1 : (0, +\infty) \mapsto \mathbb{R}$

$$f_1(\alpha) = \log \left(\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^\alpha \right)$$

and $g_2 : (0, +\infty) \mapsto \mathbb{R}$, $g_2(\alpha) = \alpha$. Then we have

$$f_1(0) = \lim_{\alpha \rightarrow 0} f_1(\alpha) = \log \left(\frac{1}{N^2} \sum_{i,j=1}^N 1 \right) = \log(1) = 0$$

as the limit exists and

$$g_1(0) = \lim_{\alpha \rightarrow 0} g_1(\alpha) = 0.$$

Both functions f_1 and g_1 are differentiable. Lastly we observe that $g_1'(\alpha) \equiv 1$. Since all the assumptions of Theorem 1 hold then

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \frac{f_1(\alpha)}{g_1(\alpha)} &= \lim_{\alpha \rightarrow 0} \frac{f_1'(\alpha)}{g_1'(\alpha)} = \lim_{\alpha \rightarrow 0} \frac{(\frac{1}{N^2} \sum_{i,j=1}^N d_{ij}^\alpha)^{-1} (\frac{1}{N^2} \sum_{i,j=1}^N d_{ij}^\alpha \log d_{ij})}{1} \\ &= \frac{1}{N^2} \sum_{i,j=1}^N \log d_{ij} = \sum_{i,j=1}^N \log(d_{ij})^{\frac{1}{N^2}} = \prod_{i,j=1}^N \log(d_{ij}^{\frac{1}{N^2}}) \end{aligned} \quad (4)$$

By Equation (4) we have the expression of Equation 3. Let

$$y_0 = \prod_{i,j=1}^N \log(d_{ij}^{\frac{1}{N^2}})$$

and we conclude by observing

$$\lim_{y \rightarrow y_0} \exp(y) = \exp\left(\prod_{i,j=1}^N \log(d_{ij}^{\frac{1}{N^2}})\right) = \prod_{i,j=1}^N \exp(\log(d_{ij}^{\frac{1}{N^2}})) = \prod_{i,j=1}^N d_{ij}^{\frac{1}{N^2}} = \sqrt[N^2]{\prod_{i,j=1}^N d_{ij}}.$$

Now we want to compute

$$\lim_{\alpha \rightarrow +\infty} Q_\alpha \quad \text{where} \quad Q_\alpha = \left(\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^\alpha\right)^{\frac{1}{\alpha}}$$

We define $d = \max\{d_{ij} | i, j \in \{1, \dots, N\}\}$ and we rewrite Q_α as

$$Q_\alpha = \left(\sum_{i,j=1}^N \frac{1}{N^2} d_{ij}^\alpha\right)^{\frac{1}{\alpha}} = \left(\sum_{i,j=1}^N \frac{1}{N^2} d^\alpha \left(\frac{d_{ij}}{d}\right)^\alpha\right)^{\frac{1}{\alpha}} = d \left(\sum_{i,j=1}^N \frac{1}{N^2} \left(\frac{d_{ij}}{d}\right)^\alpha\right)^{\frac{1}{\alpha}}$$

Next we observe that

$$\frac{d_{ij}}{d} \leq 1$$

by construction and there exist a pair (\bar{i}, \bar{j}) such that $\frac{d_{\bar{i}, \bar{j}}}{d} = 1$. Therefore it follows that

$$\sum_{i,j=1}^N \frac{1}{N^2} \left(\frac{d_{ij}}{d}\right)^\alpha = \frac{1}{N^2} \sum_{i,j=1}^N \left(\frac{d_{ij}}{d}\right)^\alpha = \frac{1}{N^2} \left(1 + \sum_{\substack{i,j=1 \\ (i,j) \neq (\bar{i}, \bar{j})}}^N \left(\frac{d_{ij}}{d}\right)^\alpha\right) \leq 1$$

for every $\alpha > 1$. And the limit in (4) is

$$\lim_{\alpha \rightarrow +\infty} d \left(\sum_{i,j=1}^N \frac{1}{N^2} \left(\frac{d_{ij}}{d}\right)^\alpha\right)^{\frac{1}{\alpha}} = d = \max_{i,j} d_{ij}.$$

35 **References**

- 36 Hill, M.O. (1973). Diversity and evenness: a unifying notation and its con-
37 sequences. *Ecology*, 54: 427-431.
- 38 Jost, L. (2006). Entropy and diversity. *Oikos*, 113: 363-375.

1 Appendix S2 - Code
2 From zero to infinity: minimum to maximum
3 diversity of the planet by spatio-parametric
4 Rao's quadratic entropy

5

6

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7 +

8 1 paRao function

```
9 function (x, dist_m = "euclidean", window = 9, alpha = 1, 1
10 method = "classic",
11 rasterOut = TRUE, lambda = 0, na.tolerance = 0, rescale =
12 FALSE,
13 diag = TRUE, simplify = 3, np = 1, cluster.type = "SOCK", 3
14 debugging = FALSE)
15 { 5
16   is.wholenumber <- function(x, tol = .Machine$double.eps
17 ~0.5) abs(x -
18     round(x)) < tol 7
19   if (!(is(x, "matrix") | is(x, "SpatialGridDataFrame") |
20 is(x,
21   "RasterLayer") | is(x, "list"))) { 9
22     stop("\nNot a valid x object.")
23   } 11
24   if (is(x, "SpatialGridDataFrame")) {
25     x <- raster(x) 13
26   }
27   else if (is(x, "matrix") | is(x, "RasterLayer")) { 15
28     rasterm <- x
29   } 17
30   else if (is(x, "list")) {
31     rasterm <- x[[1]] 19
32   }
33   if (na.tolerance > 1 | na.tolerance < 0) { 21
34     stop("na.tolerance must be in the [0-1] interval.
35 Exiting...")
36   } 23
37   if (any(!is.numeric(alpha))) {
38     stop("alpha must be a numeric vector. Exiting...") 25
39   }
40   if (any(alpha < 0)) { 27
41     stop("alphas must be only positive numbers. Exiting
42 ...")
43   } 29
44   if (method == "classic" & is(x, "RasterLayer")) {
45     isfloat <- FALSE 31
46     if (!is.wholenumber(rasterm@data@min) | !is.
47 wholenumber(rasterm@data@max) |
48     is.infinite(rasterm@data@min) | !is.wholenumber( 33
49 median(getValues(rasterm),
50   na.rm = T))) {
```

```
51         message("Input data are float numbers. Converting 35
52 x data in an integer matrix...")
53         isfloat <- TRUE
54         mfactor <- 100^simplify 37
55         rasterm <- getValues(rasterm) * mfactor
56         rasterm <- as.integer(rasterm) 39
57         rasterm <- matrix(rasterm, nrow(x), ncol(x),
58 byrow = TRUE)
59         gc() 41
60     }
61     else { 43
62         rasterm <- matrix(getValues(rasterm), ncol = ncol
63 (x),
64         nrow = nrow(x), byrow = TRUE) 45
65     }
66     message("Matrix check OK: \nParametric Rao output 47
67 matrix will be returned")
68 }
69 else if (method == "classic" & (is(x, "matrix") | is(x, " 49
70 list")))) {
71     isfloat <- FALSE
72     if (!is.integer(rasterm)) { 51
73         message("Input data are float numbers. Converting
74 x in an integer matrix...")
75         isfloat <- TRUE 53
76         mfactor <- 100^simplify
77         rasterm <- as.integer(rasterm * mfactor) 55
78         rasterm <- matrix(rasterm, nrow(x), ncol(x),
79 byrow = TRUE)
80         gc() 57
81     }
82     else { 59
83         rasterm <- as.matrix(rasterm)
84     } 61
85     message("Matrix check OK: \nParametric Rao output
86 matrix will be returned")
87 } 63
88 else ("The class of x is not recognized. Exiting...")
89 if (window%%2 == 1) { 65
90     w <- (window - 1)/2
91 } 67
92 else {
93     stop("The size of the moving window must be an odd 69
94 number. Exiting...")
95 }
96 if (np == 1) { 71
97     if (method == "classic") {
98         out <- lapply(alpha, paRaoS, rasterm = rasterm, w 73
99 = w,
```

```
100         dist_m = dist_m, na.tolerance = na.tolerance,
101         diag = diag, debugging = debugging, isfloat = 75
102     isfloat,
103         mfactor = mfactor)
104     } 77
105     else if (method == "multidimension") {
106         out <- lapply(alpha, mpaRaoS, x = x, rasterm = 79
107 rasterm,
108         w = w, dist_m = dist_m, na.tolerance = na.
109 tolerance,
110         rescale = rescale, lambda = lambda, diag = 81
111 diag,
112         debugging = debugging)
113     } 83
114     if (rasterOut == T & class(x) == "RasterLayer") {
115         outR <- lapply(out, raster, template = x) 85
116         return(outR)
117     } 87
118     else {
119         return(out) 89
120     }
121 } 91
122 else if (np > 1) {
123     if (method == "multidimension") { 93
124         stop("Multidimensional paRao not yet implemented,
125 set 'np=1'. Exiting...")
126     } 95
127     else {
128         message("\n##### Starting 97
129 parallel calculation #####")
130         if (debugging) {
131             cat("#check: Before parallel function.") 99
132         }
133         if (cluster.type == "SOCK" || cluster.type == " 101
134 FORK") {
135             cls <- makeCluster(np, type = cluster.type,
136 outfile = "",
137             useXDR = FALSE, methods = FALSE, output = " 103
138 ")
139         }
140         else if (cluster.type == "MPI") { 105
141             cls <- makeCluster(np, outfile = "", useXDR =
142 FALSE,
143             methods = FALSE, output = "") 107
144         }
145         else { 109
146             message("Wrong definition for 'cluster.type'.
147 Exiting...")
148         } 111
```

```
149         doParallel::registerDoParallel(cls)
150         on.exit(stopCluster(cls)) 113
151         gc()
152         out <- lapply(alpha, paRaoP, rasterm = rasterm, w 115
153 = w,
154             dist_m = dist_m, na.tolerance = na.tolerance,
155             diag = diag, debugging = debugging, isfloat = 117
156 isfloat,
157             mfactor = mfactor)
158         if (rasterOut == T & class(x) == "RasterLayer") { 119
159             outR <- lapply(out, raster, template = x)
160             return(outR) 121
161         }
162         else { 123
163             return(out)
164         } 125
165     }
166 } 127
167 }
```

168 2 Application of the paRao function to a syn- 169 thetic set

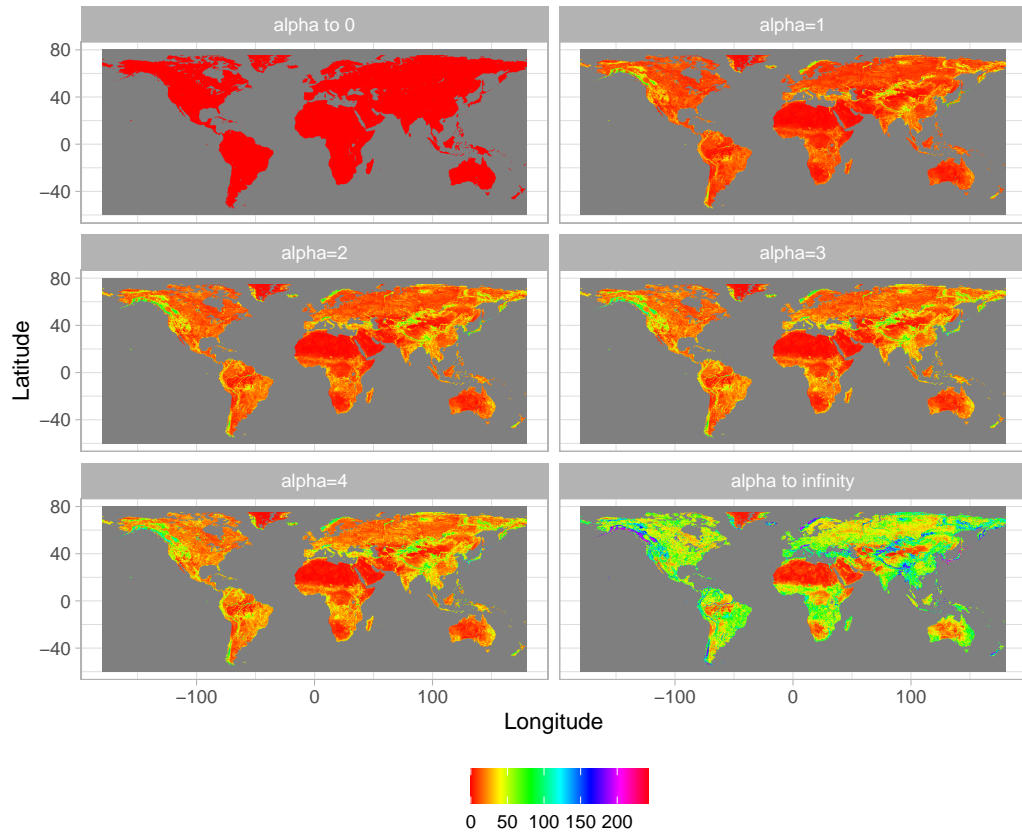
```
170 # install standalone rasterdiv
171 install.packages('rasterdiv_0.2-0.tar.gz', repos = NULL, type 2
172 = "source")
173
174 library(raster) 4
175 library(rasterdiv)
176 6
177 # generate matrix
178 synth <- raster(ncol = 8, nrow = 8, xmn = 1, xmx = 6, ymn = 8
179 1, ymx = 6)
180 values(synth) <- rpois(ncell(synth), lambda=3) 10
181
182 # paRao function, using the code in the manuscript
183 synth.parao <- paRao(synth, alpha = c(0:4,30^9), dist_m = " 12
184 euclidean", window = 9, na.tolerance = 0.5, simplify = 3,
185 diag = T, rasterOut = T)
```

186 3 Application of the paRao function to the 187 8bit copNDVI dataset

```
188 library(rasterdiv) 1
189
190 st <- paRao(copNDVI, alpha = c(0:4,Inf), 3
191   dist_m = "euclidean", window = 9, na.tolerance = 0.5,
192   simplify = 3, diag = TRUE, rasterOut = TRUE)
```

193 4 Output plot

```
194 library(raster)
195 library(ggplot2) 2
196 library(rasterVis)
197 library(RColorBrewer) 4
198
199 var.labs=c("layer.1" = "alpha to 0", "layer.2" = "alpha=1", " 6
200   layer.3" = "alpha=2", "layer.4" = "alpha=3", "layer.5" = "
201   alpha=4", "layer.6" = "alpha to infinity")
202
203 gplot(st, maxpixels=500000) + 8
204   geom_raster(aes(fill = value), color = "black") +
205   labs(x="Longitude",y="Latitude", fill="")+ 10
206   scale_fill_gradientn(colors=rainbow(100)) +
207   coord_equal()+ 12
208   theme_light()+
209   facet_wrap(~ variable, ncol = 2, labeller = labeller( 14
210     variable = var.labs))+
211   theme(legend.position = "bottom") +
212   NULL 16
```

213

1 Appendix S3 - Code for Figure 4
2 From zero to infinity: minimum to maximum
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```
7
8 library(ggplot2) 1
9 library(rasterdiv)
10 x1 <- matrix(c(255, 128, 1, 255, 128, 1, 255, 128, 1),ncol=3) 3
11 x2 <- matrix(c(10, 10, 10, 10, 50, 50, 50, 50, 50),ncol=3)
12 p1 <- paRao(x1,window=3,np=1,na.tolerance=0.1,dist_m=" 5
13 euclidean",alpha=2)
14 p2 <- paRao(x2,window=3,np=1,na.tolerance=0.1,dist_m="
15 euclidean",alpha=2)
16 alphas <- seq(0,30,1) 7
17 out1 <- paRao(x1,window=3,np=1,na.tolerance=0.1,dist_m="
18 euclidean",alpha=alphas)
19 out2 <- paRao(x2,window=3,np=1,na.tolerance=0.1,dist_m=" 9
20 euclidean",alpha=alphas)
21 r1 <- sapply(out1, function(y) {y[2,2]})
22 r2 <- sapply(out2, function(y) {y[2,2]}) 11
23 ggp <- rbind.data.frame(
24 cbind.data.frame(raop=r1,alphas,"Time frames"=rep("t0",length 13
25 (alphas))),
26 cbind.data.frame(raop=r2,alphas,"Time frames"=rep("tn",length
27 (alphas))))
28
29 pdf("landscapes.pdf") 15
30 ggplot(ggp, aes(x=alphas, y=raop,col='Time frames')) + 17
31 geom_line(size=2,alpha=0.6) +
32 geom_point(cex=3,pch=21) + 19
33 theme_bw() +
34 xlab("alpha") + 21
35 ylab("Parametric Rao") +
36 theme(axis.text.x = element_text(size=14), axis.text.y = 23
37 element_text(size=14)) +
38 theme(axis.title.x = element_text(size=16), axis.title.y
39 = element_text(size=16))+
40 theme(legend.position="top",legend.title=element_text( 25
41 size=14),legend.text=element_text(size=14))
42 dev.off()
43
44
45
46 #####
47
48 #### Second graph 31
49
50 library(raster) 33
51 library(rasterdiv) 35
52 library(ggplot2)
53
54 x1 <- matrix(c(255, 128, 1, 255, 128, 1, 255, 128, 1),ncol=3) 37
55 x2 <- matrix(c(10, 10, 10, 10, 50, 50, 50, 50, 50),ncol=3) 39
```

```
56 x3 <- matrix(c(rep(20,3),rep(250,6)),ncol=3)
57 alphas <- seq(0,30,1)
58 out1 <- paRao(x1,window=3,np=1,na.tolerance=0.1,dist_m="
59   euclidean",alpha=alphas)
60 out2 <- paRao(x2,window=3,np=1,na.tolerance=0.1,dist_m="
61   euclidean",alpha=alphas)
62 out3 <- paRao(x3,window=3,np=1,na.tolerance=0.1,dist_m="
63   euclidean",alpha=alphas)
64 r1 <- sapply(out1, function(y) {y[2,2]})
65 r2 <- sapply(out2, function(y) {y[2,2]})
66 r3 <- sapply(out3, function(y) {y[2,2]})
67 ggp <- rbind.data.frame(
68   cbind.data.frame(raop=r1,alphas,"Time frames"=rep("t0",length
69     (alphas))),
70   cbind.data.frame(raop=r3,alphas,"Time frames"=rep("tn",length
71     (alphas))))
72
73 pdf("landscapes2.pdf")
74 ggplot(ggp, aes(x=alphas, y=raop,col='Time frames')) +
75   geom_line(size=2,alpha=0.6) +
76   geom_point(cex=3,pch=21) +
77   theme_bw() +
78   xlab("alpha") +
79   ylab("Parametric Rao") +
80   theme(axis.text.x = element_text(size=14), axis.text.y =
81     element_text(size=14)) +
82   theme(axis.title.x = element_text(size=16), axis.title.y
83     = element_text(size=16))+
84   theme(legend.position="top",legend.title=element_text(
85     size=14),legend.text=element_text(size=14))
86   ggsave("~/paRao_comparison1.png",dpi=600,scale=0.5,width
87     =10,height=10)
88 dev.off()
```

